How To Be A Better Babylon Player

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Introduction
  Overview of Babylon
  Combinatorial Game Theory

Easy Cases
  Notation
  One or Two Red

Odd Sum Cases

Conclusions and Open Questions
Rules of the Game

▶ 12 tokens, 3 in each of 4 colors.
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- Two players take turns combining piles of tokens.
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- Piles can be combined if they have the same color token on top or the same height.
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- 12 tokens, 3 in each of 4 colors.
- Two players take turns combining piles of tokens.
- Piles can be combined if they have the same color token on top or the same height.
- A player wins if they are the last player to combine a pile.
Sample Game

On-line implementation of Babylon
Game Tree Analysis

- Who will win Babylon?
Game Tree Analysis

- Who will win Babylon?
- Create game tree computationally
Game Tree Analysis

- Who will win Babylon?
- Create game tree computationally
- Explore 600+ game states
- Second player if played perfectly
What is a Combinatorial Game?

- Finite.
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What is a Combinatorial Game?

- Finite.
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- No element of chance.
What is a Combinatorial Game?

- Finite.
- Two players.
- No element of chance.
- All moves known to all players.
- Babylon is impartial — all move available to all players
N and P

- Babylon is impartial — all move available to all players
- Sprague-Grundy Theorem: Any impartial combinatorial game is either \( N \) (next player wins) or \( P \) (previous player wins).
$N$ and $P$

- Babylon is impartial — all move available to all players
- Sprague-Grundy Theorem: Any impartial combinatorial game is either $N$ (*next* player wins) or $P$ (*previous* player wins).
- Example of $N$: one red and one blue
Babylon is impartial — all move available to all players

Sprague-Grundy Theorem: Any impartial combinatorial game is either \( N \) (next player wins) or \( P \) (previous player wins).

Example of \( N \): one red and one blue

Example of \( P \): three blues
(\# \text{ of colors}, \# \text{ of tokens}, \{\text{arrangement of piles}\})
Mathematical Notation - Games

- (# of colors, # of tokens, {arrangement of piles})
- The game as designed: (4, 12, {3, 3, 3, 3})
Mathematical Notation - Games

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- The game as designed: \((4, 12, \{3, 3, 3, 3\})\)
- Our focus: \((2, m, \{p, q\})\) where \(p + q = m\)
Mathematical Notation - Games

- (# of colors, # of tokens, {arrangement of piles})
- The game as designed: (4, 12, {3, 3, 3, 3})
- Our focus: (2, \(m\), \(\{p, q\}\)) where \(p + q = m\)
- Convention: red tokens least common, blue tokens most common
How Many Moves?

- How long can \((2, 5, \{2, 3\})\) last?
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- Game lasts either 3 or 4 turns
How Many Moves?

- How long can \((2, 5, \{2, 3\})\) last?
- Game ends with either one or two piles
- Game lasts either 3 or 4 turns
- If I move first, I want the game to last how long?
One Red Token With Even Number of Tokens

▶ (2, 2m, \{1, 2m − 1\})
One Red Token With Even Number of Tokens

- $(2, 2m, \{1, 2m - 1\})$
- If I start, how many moves do I want?
One Red Token With Even Number of Tokens

- $(2, 2m, \{1, 2m - 1\})$
- If I start, how many moves do I want?
- How many piles do I want?
One Red Token With Even Number of Tokens

- \((2, 2m, \{1, 2m - 1\})\)
- If I start, how many moves do I want?
- How many piles do I want?
- Winning move: cover the red token
Two Red Tokens With Even Number of Tokens

- $(2, 2m, \{2, 2m - 2\})$
Two Red Tokens With Even Number of Tokens

- \((2, 2m, \{2, 2m - 2\})\)
- If I start, how many piles do I want?
Two Red Tokens With Even Number of Tokens

- \((2, 2m, \{2, 2m - 2\})\)
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- If \(2m = 4\), cover a red with a red
Two Red Tokens With Even Number of Tokens

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Two Red Tokens With Even Number of Tokens

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Two Red Tokens With Even Number of Tokens

- $(2, 2m, \{2, 2m - 2\})$
- If I start, how many piles do I want?
- If $2m = 4$, cover a red with a red
- If $2m > 4$, cover a red with a blue
  - If opponent created a pile of size two with red on top, cover it
  - If not, cover the last red token
### Table of Possible Games

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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>3</td>
<td>4</td>
<td>5</td>
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</tbody>
</table>

Empirical investigation with Java programming

Distributed computing through Condor
Mathematical Notation - States

- Representation for current game state
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- (individual blue pile heights : individual red pile heights)
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- $(\text{individual blue pile heights} : \text{individual red pile heights})$
- Starting position $(1, 1, 1, 1 : 1, 1, 1)$
Mathematical Notation - States

- Representation for current game state
- (individual blue pile heights : individual red pile heights)
- Starting position (1, 1, 1, 1 : 1, 1, 1)
- Midgame position (3, 2 : 1, 1)
Game Tree for \((2, 7, \{3, 4\})\)
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<table>
<thead>
<tr>
<th>Node</th>
<th>Value</th>
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<tr>
<td>2,2,1</td>
<td>2,1</td>
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<tr>
<td>3,1</td>
<td>3,1</td>
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<tr>
<td>3,1,1</td>
<td>3,1,1</td>
</tr>
<tr>
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<td>3,2</td>
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Red + Blue = Odd

- $(2, 2m + 1, \{p, q\})$ where $p + q = 2m + 1$ and $p < q$
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- $(2, 2m + 1, \{p, q\})$ where $p + q = 2m + 1$ and $p < q$
- Assume we are the starting player.
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Red + Blue = Odd

- \((2, 2m + 1, \{p, q\})\) where \(p + q = 2m + 1\) and \(p < q\)
- Assume we are the starting player.
- How many moves do we want?
- We want two piles with different colors on top.
Some Important Definitions

- **Single minority pile (or SMP)**
  - only one pile of one color
  - all other piles have a height different from the above pile
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  - One pile of height 4 or greater
  - All other piles have height 1
  - The color of the minority of piles of height 1 matches the color of the pile of height 4
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- Our strategy: four-stage process
Stage 1

Place a red token on a blue token
Stage 2

- Convert a pile of height 2 to a pile of height 4
Stage 2

- Convert a pile of height 2 to a pile of height 4
- Determine the minority color
Stage 2

- Convert a pile of height 2 to a pile of height 4
- Determine the minority color
- We are now PPD!
Stage 3: PPD

- Place largest minority pile on preferred pile
Stage 3: PPD

- Place largest minority pile on preferred pile
- If single minority pile, then move to Stage 4
Stage 3: PPD

- Place largest minority pile on preferred pile
- If single minority pile, then move to Stage 4
- Otherwise, stay in Stage 3
Stage 4: SMP

Make any move that leaves the game in an SMP position
Why does such a move exist?

- Assumption: one minority pile, even number of majority piles
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- Concern: our only move creates a pile equal in size to the minority pile
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Why does such a move exist?

- Assumption: one minority pile, even number of majority piles
- Concern: our only move creates a pile equal in size to the minority pile
- If majority piles have different sizes, a move exists
- If all majority piles have the same size...
- ... then we have an even number of tokens →←
Conclusions

We have a known strategy and proof for:

- even number of tokens, one red - $N$
Conclusions

We have a known strategy and proof for:

- even number of tokens, one red - N
- even number of tokens, two red - N
Conclusions

We have a known strategy and proof for:

- even number of tokens, one red - N
- even number of tokens, two red - N
- odd number of tokens - N
Game Tree for $(2, 12, \{p, q\})$
Open Questions

- In general, is the two-color game with an even number of tokens $P$?
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- If so, what is the strategy?
Open Questions

- In general, is the two-color game with an even number of tokens $P$?
- If so, what is the strategy?
- What about more colors?