This Paper is not Complete:
A Study of Gödel’s Incompleteness Theorems

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Abstract

This paper is an introduction to the questions and issues surrounding Kurt Gödel’s famous Incompleteness Theorems, as well as an in-depth study of one example of such a Theorem. I introduce some paradoxical situations through examples such as the Liar’s Paradox and the Island of Knights and Knaves. Then the historical context of Gödel’s work is examined, progressing to a prolonged study of the incompleteness of Arithmetic. Extensive use is made of work by John W. Dawson, Jr.[2], Kurt Gödel[3], Douglas Hofstadter[4] and [5], Ernest Nagel and James Newman[6], Bertrand Russell[7], and Ramyond Smullyan[8].

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Introduction

When taking a mathematics course, whether it be high school, college, or graduate level work, we always assume one thing: the proofs and examples we study lead us to truths. In other words, there are known, simple, and valid rules of logic which govern the construction of a proof. If a proof follows these rules, then we can reliably say that the end result or final conclusion of that proof is true. The essence of writing proofs is to use our existing knowledge to gain new knowledge. It would be ridiculous to study theorems and proofs which are false (although we still write them accidentally), because this would not bring us new knowledge.

It is reasonable to believe that since all proofs that follow our fixed rules lead to truths, all true sentences must be provable. This belief is virtually forced if one thinks that proofs following our rules are the sole means of knowing what is true. Many early mathematicians held this view, up through the early 1900's. However, it is not the case that all true sentences are provable. This was proven by Kurt Gödel in 1930 in a paper titled *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*.[3] Gödel brought to a halt the world of mathematics and its vision of total axiomatization, by decisively proving the incompleteness (i.e. all truths are not provable) of arithmetic. For many years, however, people have misunderstood exactly what Gödel proved. The famous philosopher Ludwig Wittgenstein posed the question, "Does this now mean that 2+2=4.001?" [2] Fortunately, this is not the case, and 2+2 still equals 4, however Gödel's theorems point to a fundamental truth about the mechanics of mathematics.

A simple brainteaser to start with is the well-known "Liar's Paradox," credited to Epimenides of Crete. I will present a modified version here. Suppose a lawyer, Bob, makes the following statement, "All lawyers are liars." A liar is defined as someone who never tells the truth so that every statement made by a liar is false. Our brain-teasing question is, "Is Bob telling the truth?" Let's suppose that he is. Therefore his statement "All lawyers are liars" would be true. But we also know that Bob is a lawyer, and since all lawyers are liars, Bob must be a liar, and all of his statements must be false. This makes his statement "All lawyers are liars" false. But wait a second, we assumed earlier that "All lawyers are liars" was true! We have a statement which, when it is true, implies it is false.

A more concise way to present this paradox is in a sentence that refers to itself, namely "This sentence is false." If we assume the sentence is true, then this implies it is false. If we assume it is false, then this implies it is true. We could go in circles around this sentence forever. The fact that the sentence refers to itself creates the difficulty from which there seems to be no escape. A potential way out of the paradox is to set up rules governing (or forbidding) the construction of sentences that refer to themselves. These *self-referential* sentences have been popularized in recent years by Douglas Hofstadter in his books *Metamagical Themas* and *Gödel, Escher, Bach: an Eternal Golden Braid*. Some of Hofstadter's humorous and thought provoking examples of self-referential sentences are:

I am the thought you are now thinking.

There are two errors in this this sentence.
When you are not looking at it, this sentence is in Spanish.
This sentence no verb.
This sentence has cabbage six words.
If I had finished this sentence,

A self-referential paradox I have found to be particularly relevant to the inner nature of Gödel's proof concerns the concept of humility. We can define humility as not boasting about one's strengths. But when someone is truly humble, they can never tell anyone that they are humble. Why? If a person were humble, then being humble would be one of their strengths. In telling other people that they are humble, they are violating the definition of humility which says that a humble person does not boast about their strengths. Does this mean that there are no truly humble people? No, but it implies that if someone tells you "I am humble," they are lying (or just being silly).

One final introduction to these paradoxes is the Island of Knights and Knaves, a setting for puzzles commonly used by Raymond Smullyan. There are many of these mind-games and their derivatives, but we will only mention a few here. Imagine yourself on an island with two types of people, one of which always tells the truth (whom we will call Knights) and the other of which always lies (whom we will call Knaves). The problem is that these two tribes look exactly alike; there is no way to distinguish by sight whether a person is a knight of a knave. However, there are certain questions you can ask which will help you find the type of a person on the island.

Can you simply ask them what type they belong to? No. Suppose you ask a Knight, "Are you a knight?" Since Knights speak the truth, they would reply, "Yes." Suppose you ask a Knave, "Are you a Knight?" Because of Knave's compulsive lying behavior, they also would reply, "Yes." Likewise, both would reply "No" to the question, "Are you a Knave?" Instead we must ask them about their own nature of lying and truth telling, because these are things we are guaranteed about. Suppose we ask a passerby, "Do Knights tell the truth?" Only a Knight would be able to reply with a "Yes," in fact, they cannot reply with anything but "Yes," whereas a Knave would be guaranteed to tell us "No."

On this island, however, there are some truths which our assumed constraints on truth-telling prevent an islander from asserting. The sentence "I am a Knave" would be true about a Knave, but a Knave could not actually say it. A Knave who did so would have uttered a truth, which cannot be. Note, also, that a Knight who is constrained to speak only the truth can't speak "I am a Knave." Therefore under the truth telling rules for inhabitants of the island the sentence "I am a Knave" cannot be spoken.

Since we are not generally subject to the strict rules assumed of the islanders and can physically utter a sentence such as "I am a Knave," it is droll to consider an islander speculating on what to think upon meeting a person who stated "I am a Knave." Presumably, as with us, the sentence is either true or false. If it were true then the islander would infer the speaker is a Knight, in which case the sentence is false. On the other hand, if the sentence were false, then the speaker would have to be a Knave who had spoken a true sentence, an impossible combination. The net result is the need to infer that the speaker is not an islander. This is an example of an *undecidable* sentence, one that we can never know if it is true or false.

Gödel mastered aspects of self-referential statements and used them within the "island" of mathematics. He discovered a way in which some theorems and proofs in ordinary mathematics can refer to themselves. In fact, the mathematics needed, and called "ordinary" here, is merely the arithmetical system of integers using operations of addition, multiplication, etc. The vast superstructure called mathematics includes this arithmetic as its foundation. Gödel's discovery allowed him to show that there are sentences in mathematics which are true, but whose truth can't be decided by the analog of the islander, i.e. by any proof within mathematics.

Before giving Gödel's reasoning is it useful to understand the historical context in which Gödel worked. How did he come to the ambition to determine whether or not all truths were provable? To do so and to understand the logic we must develop some background about logic and axiomatic systems.

Brief History of Axiomatic Methods of Mathematics

For an understanding of the importance and impetus for Gödel's incompleteness theorems, we turn to their context in the history and philosophy of mathematics. Research into the field of logic begins with Aristotle. His main contribution was the development of the syllogism, which is the underlying structure of proof. John W. Dawson Jr. defines syllogisms in *Logical Dilemmas: The Life and Work of Kurt Gödel*[2] as

> consisting of three statements, the major and minor *premises* and the *conclusion*, each of which is of one of four forms: *universal affirmative* ("Every X is Y"), *universal negative* ("No X is Y"), *particular affirmative* ("Some X is Y") or *particular negative* ("Some X is not Y"). The premises are linked by a common "middle term" (X or Y, as the case may be) that is absent from the conclusion.

A example of a syllogism is the sequence of statements:

All P's are Q.	(universal affirmative)
S is a P.	(particular affirmative)
Therefore S is a Q.	(conclusion)

Our middle term in this example is P. It appears in the premises, but vanishes in the conclusion. To make this sound more like English rather than logic, let us substitute the word "man" for P, "mortal" for Q, and "Socrates" for S. This substitution gives us:

All men are mortal.
Socrates is a man.
Therefore Socrates is a mortal.

Aristotle found that there are fourteen forms of syllogism that produced true conclusions. This effort in the 4th century B.C. was the beginning of the foundations of logic. There was little progress in logic for 20 centuries until the late 1800's.

The next step toward clarifying logical thinking and inference was taken by Gottlob Frege, a German logician. With his books *Begriffschrift* (Concept Notation) and *Grundgesetze der Arithmetik* (Laws of Arithmetic), Frege created the first calculus of logic, and formalized the concepts of negation, implication, and quantification. He used a small number of assumed truths, or *axioms*, to generate the laws of logic and arithmetic. But Frege did not receive instant glory for his publications. Instead, they were shunned by most mathematicians of the time because of their cumbersome two-dimensional notation and cryptic symbols. Frege sacrificed understandability for exactness in his calculus of logic. He became increasingly bitter toward his colleagues until the English philosopher and logician Bertrand Russell finally recognized him as a major contributor to mathematical logic. Both Russell and Frege are founders of *logicism*, the belief that all of mathematics, every expressible property or theorem, could be reduced to logic. [7]

One of Frege's original axioms, while seemingly helping him to clarify set theory, opened himself up to a paradox. This consequence of Frege's work, named Russell's paradox because of its discovery by Russell, is best explained through the use of barbers. Imagine a town where there is one barber, and that this barber shaves only those who doesn't shave themselves. Who shaves the barber, then? If the barber does not shave himself, then because he shaves those who don't shave themselves, the barber shaves himself. But if he shaves himself, then he shouldn't shave himself, since he only shaves those who don't shave themselves. Frege had allowed a way for logic to be in the same situation as the barber, thus creating the paradox.

Around the same time as Frege's work in mathematics, Guiseppe Peano published his works concerning sets and arithmetic. His system of axioms for arithmetic, while entailing the same paradox as Frege's, was much more accessible to his colleagues, and therefore became the accepted mathematical axiom set. It is listed here in Appendix A. These axioms helped to set the study of mathematics upon the path that leads to logicism and formalism, but first we must travel through Georg Cantor.

With much research in infinity and transfinite numbers, Georg Cantor offended many of his fellow mathematicians by suggesting that infinity could be different sizes; in other words, he claimed there are different sizes of infinity, depending on what set of numbers you are talking about. We will give a brief summary of the argument here. Cantor claims that the set of real numbers is larger than the set of natural numbers. Let us suppose that they are the same, and proceed by contradiction. If they were the same size with the same number of elements, then we would be able to match up the elements one by one, such that none was left out. This approach works for the natural numbers into the even numbers: each *n* that is an element of the Natural numbers is matched up with the number 2*n* in the even numbers. Then, every number has a match. Let us presume this happens in the real numbers as well. Then, we would be able to make a list of all the real numbers proceeding from the first to infinity, and we would miss none. Cantor claimed that we can create a number from our existing set that is not in that set, and therefore we have missed that number when counting our supposed matched set. If we list our real numbers between 0 and 1 in columns, we eventually arrive at a chart such as follows:

	1 2 3 4 5 6 7 8 9.....
1	.3 3 4 5 2 1 5 7 9....
2	.8 4 1 2 6 4 7 8 9....
3	.7 5 1 4 2 3 6 9 5....
4	.2 5 4 7 6 2 5 8 4....
5	.5 4 8 7 3 1 2 5 4....
.	....

This chart extends for infinity. Cantor's missing number is created when we travel along the diagonal of this chart, extending from the point 1,1 to 2,2 to 3,3 ... *n*,*n*. If we add one to the digit, and create a new number using these digits, we see that in this example our original diagonal would be .34173… and our new number would be .45284…. .45284… is different from each of our listed numbers in at least one location, the diagonal, and this will be true for all *n* because of how we defined the creation of our new number. Therefore, if this new number is different from all the other numbers on the list, then it must not be in the list, and then it must not have been counted in the first place. In this way, Cantor showed that there were more elements in the Real numbers than in the Natural numbers, even though the number of both elements was infinity.

Cantor also researched the fields of number theory, positing his famous Continuum Hypothesis without proof. This hypothesis stated that there are no sets with a size of infinity between the real numbers and the natural numbers; all infinities could be mapped onto either one of these two options. While he proved that for many sets this was the case, he found it impossible to prove for all sets. It has since been shown that the Continuum Hypothesis cannot be proven within our normal axiomatic mathematics, but is instead a property of meta-mathematics (mathematics that talks about mathematics, rather than mathematics that talks about numbers).

These hypothesis and claims of mathematicians began to grow in their counter-intuitive nature. In response to this expansion, two main camps of mathematicians, the logicist and the formalists, plotted their goal as the cohesion of mathematics. There was also a third camp, the intuitionists, but they will not be discussed here except for this brief and dismissing sentence. If arithmetic could be shown to be fully encapsulated by a small number of axioms, then why not all mathematics? This new set of axioms for all math would provide the basis for all future mathematical study. [1]

The first camp to form was the logicists. Headed by Bertrand Russell and Alfred North Whitehead, their efforts culminated in the publication of *Principia Mathematica*, a massive three-volume work in which all the proofs involved in mathematics were reduced to logical relations. This was considered to be the finest work in mathematics for many years afterwards, and is still regarded as a highly influential work. Their project was as follows: through their notation, they reduced all the propositions and axioms of set theory to propositions of symbolic logic. Definitions were posited and constructed through logical concepts, and theorems were derived from these logical concepts following specific rules of inference. The second half of their project consisted of reducing the theories of arithmetic to set theory. With these two steps completed, the whole realm of arithmetic and mathematics could be manipulated using only logical relations, without worrying about the actual arithmetic. However, the logicists encountered some problems, namely the paradoxes of Frege and Peano, and were forced to create counter-intuitive and ultimately discarded theories concerning different levels of description (i.e. Russell's theory of types). Russell limited the scope of any formula in his system to being only able to talk about formulas on a lesser level than itself. In this manner, Russell avoids the barber paradox by saying that barbers can't shave barbers, but only non-barbers.

In response to Cantor and his contemporaries, David Hilbert began the opposite camp of formalists. He believed that mathematics was only the manipulation of symbols within a system. We can never be sure that the content of those symbols is verified, according to Hilbert's beliefs, the symbols will always be consistent. Hilbert began his career towards formalism by developing a consistent set of axioms for Euclidean Geometry. This led him to believe that all mathematics could be formalized in the same way. When invited to speak at the Second International Congress of Mathematicians on the turn of the century, Hilbert presented a list of twenty-three projects that mathematics should focus on in the twentieth century, which is listed as Appendix B. Among these are Cantor's Continuum Hypothesis, as well as the consistency of arithmetic.

To shore up the foundations of mathematics, Hilbert developed a method for proofing theorems called "proof theory," and its main goal was to show the consistency of a certain number of axioms. Proof theory consisted of listing all the primitive symbols, showing us what a good theorem was, and supplying a construction procedure. And if we knew that a set of axioms was consistent, then we would definitely know that any proof to come out of that system would be true, because consistent systems do not generate sentences which are both provable and refutable. [9]

Both of these projects, however, were doomed to failure because of Gödel's publication in a journal in 1930. In his doctoral thesis two years previous, this little known mathematician from Austria made an eternal "moment of impact" to borrow a chapter title from Dawson. Friends such as Von Neumann and others in the Vienna Circle found a new topic to debate, since apparently Gödel reopened the world of mathematics from its self-imposed constraints. [2]

First Incompleteness Theorem

Deciding how to approach Gödel's Incompleteness Theorems can be a difficult task. The resources available to you are not voluminous, but by starting in the right direction, you can save yourself many headaches and mind-cramps. The four texts researched were *On Formally Undecidable Propositions of Principia Mathematica and Related Systems* (Gödel's original paper), *Gödel, Escher, Bach: An Eternal Golden Braid (GEB)* by Douglas R. Hofstadter, *Gödel's Proof* by Ernest Nagel and James R. Newman and *Gödel's Incompleteness Theorems* by Raymond M. Smullyan.

Of these, the best place to begin in with Smullyan. I suggest first skimming through the original proof by Gödel to get an overall view of what is going to happen. Some of the symbols and notation will be unrecognizable, but it will make sense as you make progress in Smullyan. Nagel and Newman were not as helpful as I had hoped. While they eliminate much of the technical jargon, this leads to an oversimplification of the proof as well as a general impenetrable atmosphere. Hofstadter makes a great companion to reading Smullyan, with Hofstadter's enjoyable dialogues playing counter-melody to Smullyan's rigorous proofs. There are many sections of *GEB* that are not particularly relevant to Gödel's proofs, but they provide hours of recreational thinking. In this paper, I will lead you through Smullyan's path to his first proof of incompleteness.

Smullyan's introduction to the proofs begins with a puzzle on page 2 concerning printability and computers. Using five symbols, ~ *P N* ( ), he easily constructs a formal system in which an unprintable sentence can be found. An *expression* can be any string of these symbols, ex. *PN*~((*N*)~*P*, or *P*~*N*~*P*~*N*. A *sentence* is one of the following forms, *P*(X), *PN*(X), ~*P*(X), and ~*PN*(X), where X stands for any expression. Sentences are usually defined as the meaningful set of expressions, and this holds for our current example, with *P*(X) standing for X is printable, and so on. It is important to keep in mind that the symbols *P*, ~, and *N* stand for "printable" and "the norm of" as well as being symbols themselves. The sentence *P*(*P*~*N*()) would be interpreted as "the expression *P*~*N*() is printable by the machine."

We are told that the machine is always correct in its printing, i.e. every sentence printed will be a true sentence. And truth is defined as if a sentence asserts something is printable, then that thing is actually printable. But Smullyan quickly points out that this machine cannot print all true sentences. For example, one of the sentences it cannot print is "~*PN*(~*PN*)" because this statement says "It is not the case that the norm of ~*PN* is printable." Let us assume it is true and can be printed. What is the norm of ~*PN*? by his definition of norm, it is ~*PN*(~*PN*), the very statement we just printed. So this statement asserts that it is unprintable.

In the same manner, we can understand the second puzzle. With the symbols ~ *P N* 1 0, he introduces us to the concept of Gödel numbering. Each symbol of the language is assigned a number: ~ = 10, *P* = 100, *N* = 1000, 1 = 10000, and 0 = 100000. Norm is redefined to be an expression followed by its Gödel number. We can see that each Gödel number is a printable expression of our machine. The norm of ~*P*~01*P* would be ~*P*~01*P*101001010000010000100. Our challenge is to again find a sentence that is true but not printable by this machine. Smullyan's solution is ~*PN*101001000. Why? The sentence ~*PN*101001000 asserts the non-printability of the norm of the expression denoted by the Gödel number 101001000. The expression denoted by 101001000 is ~*PN*, and the norm of ~*PN* is ~*PN*101001000. This sentence again states that it is unprintable.

Smullyan now moves forward onto what he calls an Abstract Form of Gödel's and Tarski's Theorems. This is the most important section of the book. Here, Smullyan sets up the language that he uses throughout the remaining chapters; grasping these few pages is essential to understanding his explanation of Gödel's proof.

While his explanations of the language L are very clear (see [8] pg. 5), it is helpful to see a picture representation of how the language fits together:

We see that while expressions of $\boldsymbol{H}$ are not necessarily sentences, expression such as $H(n)$ are required to be a sentence. This combination of predicates and natural numbers using the function Φ is particularly important to the development of a self-referential sentence.

The concept of truth is introduced here as well, but with lack of an explanation, we are unsure of what makes a sentence true or false. While this is an important concern to have, it will not become an issue until we define our axiom system later. For now, it is only important to understand the notion of expressibility. Remember that the Φ function is assigned for every predicate H and every number n. Therefore, whenever n causes H to be true, we can say that n belongs to a certain set A according to the relation:

$$H(n) \in \boldsymbol{T} \Leftrightarrow n \in A$$

We must also keep in mind the definition of correctness. The proofs to follow rely upon the assumption that our language is correct, and therefore involves an undecidable sentence. As long as there are no provable or true sentences which are refutable and the provable sentences are all true, there will be a sentence which is true but not provable.

We now define Gödel numbering, the next important concept in understanding the proofs. We assign to every expression in E (all combinations of symbols in L) a specific natural number according to a function g. The important thing to understand the assignment of a number to an expression; E would get the number $g(\mathrm{E})$. And if we say that n is the Gödel number for an expression, we would say this by writing E_n. Then $g(\mathrm{E}_n) = n$. For more concrete examples of Gödel numbering, see Appendix C.

With the additional concept of diagonalization (the expression $E_n(n)$) we are almost ready to construct our undecidable sentence. Recall that the Φ function is assigned for every expression, and when E_n is a predicate, $E_n(n)$ will be a sentence, which will be true in the case that n causes E_n to be true. But you may ask, why is it called diagonalization? As in the diagonalization argument by Cantor, we make a chart with the expressions of form E_n going horizontally, and numbers n going vertically. If we order n the same way in both directions, the diagonal down the chart will be the expressions of the form $E_n(n)$. It may be easier to understand the diagonal function $d(n)$ if we expand it as follows:

$$d(n) = g(E_n(n)) = g(E_n(g(E_n))$$

i.e. we give each expression a number n, then when the expression E_n is a predicate, we use n to create a sentence $E_n(n)$, and the number of this expression is written as $d(n)$.

We can see that not every number n will give us a number $d(n)$, because not every expression is a predicate. But for any set A, we can let the set A* be the set of all numbers n such that $d(n)$ is an element of A. This way, we can talk about the set of numbers which can create Gödel numbers $d(n)$. If we add the concept of compliments (~A is the set of all numbers which are not in the set A) we now have all the building blocks ready to prove the incompleteness of our language L.

Smullyan's exposition of the proof is very thorough, but some of us could use help remembering all the specific definitions we learned along the way. First, let us address the issue of $\sim P^*$. This set is the set of all numbers which are not members of P^*, and P^* is the set of all numbers n such that $d(n)$ is an element of P. Since P is the set of all Gödel numbers of provable sentences, P^* is the set of all numbers n such that the Gödel number of the diagonal of the expression assigned to n is the number of a provable sentence. So a number which is in $\sim P^*$ would be one who's diagonal number is not the number of a provable sentence.

Our first assumption is that $\sim P^*$ is expressible in L. This means by the definition of expressible that there is a predicate that expresses $\sim P^*$. Let us call this expression H, and let the Gödel number of H be h. Then if we follow Smullyan's reasoning, we see that $H(n)$ is true $\Leftrightarrow n \in \sim P^*$, and therefore $H(h)$ is true $\Leftrightarrow h \in \sim P^*$. Now, looking back at our definitions, we see that h is an element of $\sim P^*$ iff $d(h)$ is an element of $\sim P$, and $d(h)$ is an element of $\sim P$ iff $d(h)$ is not an element of P. Since P is the set of Gödel numbers of provable sentences, then $d(h)$ must not be the number of a provable sentence. $d(h)$ is the Gödel number for the expression $H(h)$, and therefore this implies that $H(h)$ is not provable. In symbol notation:

$$H(h) \text{ is true} \Leftrightarrow h \in \sim P^* \Leftrightarrow d(h) \in \sim P \Leftrightarrow d(h) \notin P \Leftrightarrow H(h) \text{ is not provable.}$$

This means one of two things: either $H(h)$ is true and not provable, or $H(h)$ is false and provable. Since we know by assumption that L is correct (all provable sentences must be true) it is obvious that it must be the case that $H(h)$ is true and not provable. We have found our undecidable sentence, and this undecidable sentence shows us that our language *L* is incomplete.

For all subsequent proofs for specific languages, the same procedure will be used to prove the incompleteness of that particular system. Since this incompleteness is reliant on the expressibility of the set ~*P**, it will be necessary to demonstrate the existence of ~*P**. This is done through Smullyan's three propositions G_1, G_2, and G_3. The combination of these propositions ensures us that ~*P** will be expressible in our language *L*.

The rest of section I illustrates other results using our definitions and assumptions. The Diagonal Lemma and Theorem (T) follow immediately from the definitions and show another way to manipulate the definitions to arrive at an undecidable sentence. But the most important concept to understand is the Gödel Sentences. We see that a sentence E_n is a Gödel sentence for a set A if either E_n is true and *n* is in A, or E_n is false and *n* is not in A. As Smullyan says, it is a sentence which asserts that its Gödel number is in A.

Section II begins by defining some of the terms we have used already, but with more specific definitions. Consistent vs. inconsistent and complete vs. incomplete are both applied to languages, while decidable and undecidable are applied to individual sentences. In the Dual Theorem, we reverse the procedure, such that instead of working with a sentence which is true, we are instead working with a theorem that is asserts it is false but not refutable. The same proof technique is used as before, only with the goal being non-refutability rather than non-provability. Rather than repeat the Dual Theorem here, we will instead explain a proof of Exercise 1 on page 12.

We are given that *L* is correct,(1) *P** is expressible in *L*, and (2) for any predicate *H*, there is a predicate *H*' such that for every *n*, the sentence *H'*(*n*) is provable in *L* iff *H*(*n*) is refutable in L. First, we assume the premises. Since *P** is expressible in *L*, let *H* be the predicate which expresses *P**, and *h* be the Gödel number of *H*. By condition (2), we know there exists a predicate *H*' and we will let *h*' be the Gödel number of *H*'. Because *H*(*n*) is true $\Leftrightarrow n \in P^* \Leftrightarrow d(n) \in P$ for all *n*, (from our definition of expressibility and the set A*) we can say that it will be true for a particular *n*, such as *h'*. Therefore *H*(*h'*) is true $\Leftrightarrow d(h') \in P$. If *d*(*h*') is an element of *P*, then this means that the expression *H'*(*h*') is a provable sentence, because of the definition of the diagonal function. Now that we know that *H'*(*h*') is a provable sentence, by condition (2) we see that *H'*(*h*') is provable $\Leftrightarrow$ *H*(*h*') is refutable. Therefore *H*(*h'*) is true $\Leftrightarrow$ *H*(*h*') is refutable. This again leaves us with two options, either *H*(*h*') is true and refutable, or *H*(*h*') is false but not refutable. Since *L* is correct, it must be the case that *H*(*h'*) is false but not refutable. Because *H*(*h*') is false, it is not provable, either, and therefore *H*(*h*') is an undecidable sentence in *L*.

Another valuable exercise to solve is Exercise 6. Is the set ~A* the same as the set ~(A*)? A Venn diagram can best show the answer to this:

The set ~A* will give us the numbers which are Gödel numbers of diagonal sentences in ~A, while ~(A*) will give us the numbers which are not Gödel numbers of diagonal sentences in A. We see that any element which is an element of A but not of A* is a member ~(A*) but not of ~A*. Because of this possibility, these sets are not identical.

As we proceed into the concrete example of Gödel's incompleteness theorem for Peano Arithmetic, it is helpful to keep in mind the structure of how the proof will proceed. First, we will identify variables and functions within our language. Variables and functions then combine to form terms, and formulas are logical combinations of terms. Arithmetic sets are constructed in such a way to be expressed by our formulas. Our specific functions, such as concatenation, power, Gödel numbering, and formation sequences are all shown to be Arithmetic and thus definable via a Gödel number. The Axioms of Peano Arithmetic are then show to be Arithmetic, as well as the inference rules of modes ponens and derivability. Proofs are then constructed with these axioms and inference rules, and shown to be arithmetic as well. Then we are finally able to construct the set *P* and *R* of the Gödel numbers of provable and refutable sentences. Once we have the sets *P* and *R*, we are able to express ~*P**, giving us the pieces necessary for the incompleteness.

Smullyan begins chapter II with explaining the symbols and their uses in the language L_E, all except #. This symbol is explained in chapter III, and it is used to separate formulas in a proof sequence, but this information will not be needed until later. It is helpful now to see some examples of ordinary formulas written in the language L_E.

2 + 2 = 4 -- $0''f_{,}0''=0''''$

4 * 3 = 12 -- $0''''f_{,,}0'''=0''''''''''''$

2x +4y = 9z -- $0''f_{,,}(v_{,})f_{,}0''''f_{,,}(v_{,,})=0'''''''''f_{,,}(v_{,,,})$

It is important to note that all our definitions of formulas are defined by induction. Each is built from the previous definitions and adds a new degree onto the depth of the function. Later, our definitions of terms will be defined recursively, but the end result is the same. Induction gives us a more accessible explanation, while the recursive lets the functions be more easily manipulated as elements of the language.

The logical formulas are described on page 17, showing us exactly how each logical formula of first order logic can be constructed using only negation (~), implication ($\supset$), and the universal quantifier ($\forall$). We should also keep in mind that since each formula is necessarily constituted of atomic formula, and these are necessarily either of the form $t_1 = t_2$ or $t_1 \leq t_2$, then all of our formulas are relations between equalities or inequalities of numbers. Smullyan's definitions of truth in our formulas appear to be self-evident, but it is important to specifically define them, rather than depending on an intuitive notion of truth.

Arithmetic sets are analogous to our previous discussion of predicates. For the extent of this paper, we will be dealing with Arithmetic sets rather than arithmetic sets (i.e. sets that are expressible in L_E rather than sets only using plus and times.) A set is Arithmetic if it is expressed by some formula of L_E, and this becomes vitally important to our proof of incompleteness. If a set is arithmetic, then there is a formula for which all numbers in that set cause the formula to become true, and these formulas can be written in the symbols of our language. We can see that the set "x divides y" as described in Exercise 2 can be shown to be arithmetic by the formula $\exists v_2\ (x * v_1 = y)$. The set "$x$ is prime" is slightly more complicated, being represented by the formula: $(\sim\exists y)((y\neq x)\wedge(y\neq 0')\wedge(\exists z)(y * z = x)$.

Smullyan's next step is to introduce concatenation as a function describing the combination of numbers. For example, the number 45278 is the concatenation of the numbers 45 and 278 (and also the concatenation of 452 and 78). We can formalize this function by having the first number be multiplied by 10 to the power of the length of the second number, and then added to the second number. When our second number is 278, the length of this (how many digits there are) is three. 10 to the power of three is 1000, and this multiplied by 45 gives us 45000. 45000 plus 278 gives us 45278, and this was our goal number. Smullyan also demonstrates how this concatenation can be formalized in any base greater than two. He explains the technical procedures necessary to show that the concatenation procedure is Arithmetic, but I believe it is more important to understand the general idea that a number can be seen as a combination of numbers. This is contrary to our usual understanding of combination (we usually add 452 and 78 to get 530) but necessary for our discussion of Gödel numbering.

As we saw in the examples concerning printability, Gödel numbers are numbers which stand for an expression or symbol. More formally, Gödel numbers are the concatenation of the numbers of individual symbols. Since Arithmetic expressions or sentences talk about numbers using the language L_E, through assigning each symbol a Gödel number, it is now possible construct expressions which talk about expressions by referring to their Gödel number. But exactly how does this work? Four systems of Gödel numbering are listed in Appendix C and this provides a good comparative basis to understand the abstract mechanics of Gödel numbering. However, as more examples are always helpful, I will provide them for you here.

Presumably, the formula for concatenation, since it is Arithmetic, can be written in the language L_E. Therefore we could assign a number to the formula of concatenation through the concatenation process of Gödel numbering. Let us take an easier example.

$2 + 2 = 4$ ---- $0''f_{,}0''=0''''$

The Gödel number for this expression would be: 10045100η1000 (in base 13 notation).

$2x + 4y = 9z$ ---- $0''f_{,,}(v_{,})f_{,}0''''f_{,,}(v_{,,}) = 0'''''''''f_{,,}(v_{,,})$

This expression’s Gödel number being: 100455265345100045526553η1000000000455265553
We can see how easy these expressions can generate enormous and cumbersome numbers, and we are lucky that these numbers are never actually present in the proof. What is used is substitution of these numbers for variables, such as n, which are much easier to manipulate.

The final section of chapter II involves the function $r(x,y)$. An equivalence is drawn between an expression $F(\boldsymbol{n})$ and an expression $\forall v_1 (v_1 = \boldsymbol{n} \supset F(v_1))$. It is much easier to explain this through words than through symbols, though. If we wish to find the expression $F(v_1)$ where we substitute $\boldsymbol{n}$ for $F(v_1)$, what do we want to substitute in for v_1? If we have a set of objects which are possible to substitute, the only ones which will guarantee us the result of $F(\boldsymbol{n})$ are those equal to $\boldsymbol{n}$. This is the basic principle behind our equivalence. Smullyan lets the expression $\mathrm{E}[\boldsymbol{n}]$ stand for $\forall v_1 (v_1 = \boldsymbol{n} \supset F(v_1))$. Why do we need to make this substitution in the first place? As he says, the formula $\forall v_1 (v_1 = \boldsymbol{n} \supset F(v_1))$ can be shown to be an Arithmetic formula, and it is constructed from the Gödel number of $F(v_1)$ and the number $\boldsymbol{n}$. This enables our Arithmetic function to be easily reducible to other functions which such as concatenation, something that has already been shown to be Arithmetic. When we can show that the set P of Gödel numbers of provable sentences is Arithmetic through this gradual building process, we will be able to complete our proof.

We let the function $r(e,n)$ stand for the Gödel number of our expression $\mathrm{E}[\boldsymbol{n}]$, where e is the Gödel number of the expression E. Then we can generalize this to all expressions $\mathrm{E}_x[y]$ with the function $r(x,y)$. What does this number look like? It will consist of $965276\eta * 13^y * 8 * x * 3$. This is a perfectly Arithmetic function, with the only variables being the x and y of our numbers. Now we add our twist. Since $r(x,y)$ is Arithmetic for any values of x and y, then it will obviously be Arithmetic if we set $y = x$, giving us the function $r(x,x)$. This is the Gödel number of the expression $\mathrm{E}_x[x]$ (does this look familiar from the Abstract proof?) Smullyan sets $d(n)$ equal to $r(x,x)$, and then recreates the set A^*, only now it has an actual Arithmetic basis. This demonstrates that A^* is a set within our language L_E, fulfilling our Condition G_1 from page 7. Also, Smullyan here states that Condition G_2 is fulfilled as well. The complement of a set A which is expressible is given by the negation of the function which expresses that set A. “If $F(v_1)$ expresses A, then its negation $\sim F(v_1)$ expresses the complement $\sim A$ of A.”

The final section of Smullyan’s book we will look at is Chapter III: The Incompleteness of Peano Arithmetic with Exponentiation. He lists 19 axioms of propositional logic and standard arithmetic with exponentiation, each one being written in either the language L_E or abbreviations of that language. We can see the parallels here between his axioms and the listing of Peano’s original axioms in Appendix A. Axiom N_2 states that zero is not the successor of any number, axiom N_5 states that any number multiplied by zero will be zero, and so on. The only potentially confusing axiom is the single axiom of Group IV. This is what Smullyan calls the Axiom of induction. If we recall the formation of the expression $\mathrm{E}[\boldsymbol{n}]$ as equal to the formula $\forall v_1 (v_1 = \boldsymbol{n} \supset F(v_1))$, we see that the axiom of induction is a double substitution into this formula. First we substitute v_1 for all v_{i}, and then we substitute v_1' for v_1. In English this axiom states that if we have a function which is true for the numeral denoted by 0, then if for all v_1, if the function is true for v_1, then it is true for an equivalent function F[v1]. This notion of the induction axiom bypasses the notion of free variables and substitution, rather we replace them rather than substitute them.

A proof (a very important concept if we are to assert that things are provable) is defined as a sequence of formulas in which each is either an axiom, or directly follows from previous formulas by way of Rule 1 and Rule 2, in which the Rules are the standard inference rules of Modus Ponens and Generalization. The last sentence in such a sequence is considered to be proven by the previous sentences to be true, since we assume that the axioms are true, and our rules of inference can only produce true results.

But unfortunately we are not yet ready to prove the incompleteness of the language L_E. We first must create more functions to properly describe the processes involved in a proof. Smullyan creates three functions, *xBy*, *xEy*, and *xPy*, which stand for *x* begins *y*, *x* ends *y*, and *x* is a part of *y*, where *y* is a sequence of numbers. These are shown to be Arithmetic, with each building from the previous definitions. Before we proceed, a point needs to be made concerning the remark on page 31. If you are planning to continue researching these Theorems on your own and wish to finish Smullyan's book, then it will be necessary to pay attention to the remark. Otherwise, you may ignore it, and think of the expressions $(\exists z \leq y)$ as only $\exists z$.

Why is it necessary to constantly create new functions, such as the before and after functions? When we are attempting to show that proofs can be Arithmetized (i.e. expressed by some formula of the language L_E) then it becomes necessary to convert them into their Gödel numbers. Through these functions of before, after, and part of, we can accurately pick out certain sentences in the proof. As I stated earlier, the symbol # would become relevant when we begin to combine our expressions into sequences which form a proof. These sequences take the form of $\#X_1\#X_2\#.......\#X_n\#$, where each expression is enclosed between two #'s, and each expression does not contain a #. #'s are reserved for separating segments of the sequences, and if there are random #'s in our formulas, then it will not be a well-formed proof (wff). As before, it is possible for these poorly formed formulas to exist in our language, but we are not concerned with them, since they have no truth-value.

We continue with our production of functions by adding Seq(*y*), $x \in y$ and $x < y$, which stand for *y* is a sequence, *x* is a member of a sequence *y*, and element *x* is earlier than element *y* in a sequence *z*. With only one more step necessary, we are almost ready to proceed to the incompleteness theorem, but first we must reformulate our definitions of terms and formulas. Earlier, our definitions were only inductive and vague, but now we will use the explicit recursive functions to illustrate our terms and formulas, thus making them Arithmetizable. These formulas use the same principles and same functions as our sequences of formulas in a proof. Now all the pieces are in place like a set of dominoes in a line, and all we must do is set them in motion.

Pages 34 and 35 can be compared with the middle section of Gödel's original proof, but whereas Gödel used 43 steps, Smullyan can complete the job in 17. Admittedly, Smullyan uses a larger notation than Gödel, but also, Smullyan has the experience of Gödel, Carnap, Tarski, and many others to build upon to make this the shortest incompleteness theorem around.

First, we show that a string of subscripts (the symbol ,) can be Arithmetized by using only symbols of the language $L_{E.}$ Next, we create variables by combining subscripts with the symbols for “(v” and “)”, 26 and 3. A numeral is defined in straightforward terms, something made possible by Smullyan’s manipulation of the Gödel numbers of the prime symbol and 0. Had this relationship not been artificially created. It still would have been possible to create a function to describe numerals, but it would have been much more difficult to conceptualize. Our relation of terms is created next, and followed by the formation sequence of terms, and then the definition of a term itself. We can begin to notice that every definition is either explicitly defined using our language, or else is defined in terms of the previous formula.

Atomic formulas are shown to be composed of terms in equality or inequality relationships, and then relations of terms are defined to be our symbols of implication, negation, and generalization. Sequences of formulas follow, as well as normal formulas themselves, again, each being defined from the previous definitions. Each Axiom of Peano Arithmetic is then treated individually and converted into symbol notation using formulas as its building blocks. Axioms describe the relationship of formulas to one another, and therefore since formulas are Arithmetizable, so are the Axioms.

The inference rules of Modus Ponens and Generality combine to create the derivable function. We must keep in mind that because of our Gödel numbering, we are not actually talking about the formulas, but we are talking about the numbers that represent the formulas in our language. It is easy to lose sight of this, but we must remember that all we are describing are functions that have numbers and only numbers as their input and output. Finally, a sequence of derivable formulas are shown to be a proof, and from here, it is a very easy step to determining whether or not a certain formula is the end of a sequence of provable statement, and thus the formula is provable within the language. We have now shown the systematic process by which we arrive at the set of Gödel numbers of provable sentences. Now we have fulfilled Condition G_1, completing the process begun at the end of Chapter I. Using our previously defined sets, we know that since P is expressible, that P^* is expressible, and $\sim P^*$ is expressible as well. And if $\sim P^*$ is expressible in the language L_E, then we can apply our argument concerning the abstract proof to this specific instance, finding a specific formula which is true but not provable.

Smullyan states that this is the simplest incompleteness theorem he knows of. Further chapters in his book examine arithmetic without the exponential function, as well as an incompleteness theorem based on a much more primitive axiomatic system. I believe that an amateur logician can master the proofs, if the overall goal of the theorems is always kept in mind. It is easy to become overly focused on the technical aspects of the proof and miss the overall coherence of each part. Yet at the same time, it is also an easy approach to totally dismiss the technical and rigorous aspects, and this is in error as well. Both approaches are necessary to gain a complete, rather than incomplete, understanding of Gödel’s Incompleteness Theorems.

Appendix A

Axioms of Zermello-Frankel Set Theory

1. A = B iff $A \subset B$ and $B \subset A$

   $(\forall x)\ (x \in A \supset x \in B) + (\forall y)\ (y \in B \supset y \in A)$

2. If P is a property and A is a set then

   $\{x \in A \mid P(x)\}$

3. If A and B are sets, then there is a set C such that $x \in C \Leftrightarrow (x=A \text{ or } x=B)$

   Def $C = \{A,B\}$

4. If A is a set, then there is a set Ω such that $x \in \Omega \Leftrightarrow x \in X$ for at least one X in A.

   Def $\Omega = \bigcup_{X \in A} X$

5. If A is a set then there is a set B such that $X \in B \Leftrightarrow X \subseteq A$

6. If P is a single valued relation whose first elements form a set A, then the second elements form a set A

7. Axiom of Choice. If A is a set of non-empty sets then $\exists \Psi$ function defined on elements X in A such that $\Psi(X) \in X$ Target set of $\Psi = \bigcup_{X \in A} X$

8. Axiom of Infinity. There is at least one infinite set.

Axioms of Peano Arithmetic
(*as found in Russell's Introduction to Mathematical Philosophy* [7])

1. 0 is a number
2. The successor of any number is a number
3. No two numbers have the same successor.
4. 0 is not the successor of any number.
5. Any property which belongs to 0, and also to the successor of every number which has that property, belongs to all numbers.

Appendix B

Hilbert's Twenty-Three Problems of Mathematics

1. Cantor's Problem of the Cardinal Number of the Continuum
2. The Compatibility of the Arithmetical Axioms
3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes
4. Problem of the Straight Line as the Shortest Distance Between Two Points
5. Lie's Concept of a Continuous Group of Transformations Without the Assumption of the Differentiability of the Function Defining the Group
6. Mathematical Treatment of the Axioms of Physics
7. Irrationality and Transcendence of Certain Numbers
8. Problems of Prime Numbers
9. Proof of the Most General Law of Reciprocity in any Number Field
10. Determination of the Solvability of a Diophantine Equation
11. Quadratic Forms With Any Algebraic Numerical Coefficients
12. Extension of Kronecker's Theorem on Abelian Fields to Any Algebraic Realm of Rationality
13. Impossibility of Solution of the General Equation of the 7th Degree by Means of Functions of Only Two Arguments
14. Proof of the Finiteness of Certain Complete Systems of Functions
15. Rigorous Foundations of Schubert's Enumerative Calculus
16. Problem of the Topology of Algebraic Curves and Surfaces
17. Expressions of Definite Forms by Squares
18. Building up of Space From Congruent Polyhedra
19. Are The Solutions of Regular Problems in the Calculus of Variations Always Necessarily Analytic?
20. The General Problem of Boundary Values
21. Proof of the Existence of Linear Differential Equations Having a Prescribed Monodromic Group
22. Uniformization of Analytic Relations by Means of Automorphic Functions
23. Further Development of the Methods of the Calculus of Variations

-taken from **Math Odyssey 2000** at http://loki.sonoma.edu/Math/faculty/falbo/hilbert.html

Appendix C

Versions of Gödel Numbering

In my research, I have found four versions of Gödel numbering used by various authors [3], [5], [6] and [8]. Each is equally useful in constructing an incompleteness proof. There are infinite possibilities for creating a Gödel numbering. The numbering chosen for the bulk of this paper is due to Smullyan[8].

On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems

Kurt Gödel (1931) [3]

There are seven symbols in Gödel's numbering, and each is assigned an odd number.

0	-	1	[the number zero]
f	-	3	[the successor function]
~	-	5	[negation]
∨	-	7	[disjunction]
Π	-	9	[universal quantifier]
(	-	11	[left parenthesis]
)	-	13	[right parenthesis]

Quoting Gödel, "variables of type n are given numbers of the form p^n (where p is a prime number > 13)." To create the Gödel number of a sentence, we first translate all the symbols into their respective numbers, and place these in a list, or "series." Suppose this list is n_1, n_2,. . . n_k. Then the list is associated a natural number by "letting the number $2^{n_1} \cdot 3^{n_2} \cdot \ldots \cdot p_k^{n_k}$ correspond to the series n_1, n_2, . . . n_k , where p_k denotes the k-th prime number in order of magnitude." For example, to derive the Gödel number for the sentence $\mathbf{0} \vee \sim \boldsymbol{f}\, \mathbf{0}$, we first make a list of the numbers associated with the symbols: 1,7,5,3,1. The Gödel number of the sentence is then given by the number: $2^1 \cdot 3^7 \cdot 5^5 \cdot 7^3 \cdot 11^1$, or 51572193750. "A natural number is thereby assigned in one-to-one correspondence, not only to every basic sign, but also to every finite series of such signs." The reader can see that Gödel is using the Fundamental Theorem of Arithmetic which asserts that every integer > 1 factors uniquely (apart from order) into a product of powers of primes.

Gödel's Proof

Ernest Nagel and James B. Newman (1958) [6]

Nagel and Newman use a language with ten symbols as opposed to Gödel's original seven. They use the same techniques as Gödel, but have expanded the language for clarity.

~	-	1	[negation]
v	-	2	[disjunction]
⊃	-	3	[implication]

∃ - 4 [existential quantifier]
= - 5 [equals]
0 - 6 [the number 0]
s - 7 [the successor function]
(- 8 [left parenthesis]
) - 9 [right parenthesis]
, - 10 [comma]

Numerical variables (for which a number can be substituted) are assigned a prime number greater than ten and it depends on what you use as to how you assign them. Sentential variables (for which formulas may be substituted) are assigned a prime greater than 10^2. And predicate variables (for which predicates such as *Prime* and *Greater Than* may be substituted)

The Gödel number of a sentence is obtained in the same way as the previous system. For example, the expression **(∃ x) (x = s y)** is assigned the list of numbers 8,4,11,9,8,11,5,7,13,9. This list is then put into our combination of primes formula: $2^8 \cdot 3^4 \cdot 5^{11} \cdot 7^9 \cdot 11^8 \cdot 13^{11} \cdot 17^5 \cdot 19^7 \cdot 23^{13} \cdot 29^9$, or 145666408161709409197789938288649818781891470181481887898950349321995516094737500000000 ≈ (1.46 x 10^{86})!

Gödel's Incompleteness Theorems

Raymond M. Smullyan (1992) [8]

Smullyan uses a language with thirteen symbols to create his Gödel numbering. He uses Quine(1940) as a model, but where Quine used nine symbols, Smullyan has thirteen.

0 - 1 [the number zero]
′ - 0 [the successor function]
(- 2 [left parenthesis]
) - 3 [right parenthesis]
f - 4 [function]
, - 5 [subscript]
v - 6 [variable]
~ - 7 [negation]
⊃ - 8 [implication]
∀ - 9 [universal quantifier]
= - η [equals]
≤ - ε [less than or equal to]
- δ [series divider]

Smullyan makes use of base 13 arithmetic to express Gödel numbering. He uses the Greek letters η, ε, and δ to denote 10, 11 and 12 in the base thirteen notation. To obtain Gödel numbers, Smullyan simply replaces each symbol with its corresponding numerical digit, and this string is our base 13 Gödel number. For example, the expression **~ (0′ = 0)** would be assigned the number 7210η13. This notation is more straight-forward than either Gödel or Nagel & Newman, and this in turn simplifies the incompleteness proof.

Gödel, Escher, Bach: An Eternal Golden Braid

Douglas R. Hofstadter (1979) [5]

Where Nagel and Newman's notation was an expansion of Gödel's numbering system, Hofstadter's notation can be seen as an expansion of Smullyan's system, although Hofstadter's was written thirteen years earlier. There are 21 symbols in Hofstadter's language, called Typographical Number Theory (TNT).

0	-	666
S	-	123
=	-	111
+	-	112
·	-	236
(	-	362
)	-	323
<	-	212
>	-	213
[	-	312
]	-	313
a	-	262
′	-	163
∧	-	161
∨	-	616
⊃	-	633
~	-	223
∃	-	333
∀	-	626
:	-	636
punc.	-	611

Hofstadter then combines the numbers in the same manner as Smullyan. The expression ∀**a:(a+0)=a** is translated into the number 626,262,636,362,262,112,666,323,111,262.

We can see that in every Gödel numbering system, there are trade-offs that must be made. Gödel's original notation is small with only seven symbols. While this makes the numbering somewhat easier (i.e. less symbols means less numbers), the overall notation is difficult to understand. All the theorems and sentences of arithmetic can be written in this minimalistic notation, but the constructions necessary for even the simplest formulas are enormous. On the other extreme, the 21 symbols of Hofstadter’s TNT are seemingly more expressive and are easily manipulated, but the numbering system is required to be a tangle of 1,2,3, and 6.

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