Math 105: Finite Mathematics

Study Guide

February 21, 2008

1 Logical Formulas

In logic, there are **statements** which can be true or false. We typically abbreviate these statements with either one or a few letters, such as

- $P =$ "I am 20 feet tall."
- $Q =$ "The moon is made of green cheese."
- $R =$ "The sky is blue."

and use **T** and **F** to represent True and False.

Basic statements are combined into more complicated statements using four main connecting **operators**: $\neg$ for NOT, $\vee$ for OR, $\wedge$ for AND, and $\Rightarrow$ for IMPLIES. They have the following truth tables. For each statement, we consider that it might be T or F, two possibilities, and so for n statements, we must consider 2^n possible worlds.

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

2 Tautologies and Contradictions

To evaluate a particular compound statement, we usually go and find the truth value of each individual statement, then look up the proper entry in the truth table. However, there are some compound statements where this information is irrelevant. They are called **tautologies** when the statement is true in all possible worlds, and **contradictions** when the statement is false in all possible worlds.

An example of a tautology is $(P \Rightarrow (Q \vee R)) \vee (\neg Q \wedge \neg R)$:

P	Q	R	$(P$	$\Rightarrow$	$(Q \vee R))$	$\vee$	$(\neg Q$	$\wedge$	$\neg R))$
T	T	T		T	T	**T**		F	
T	T	F		T	T	**T**		F	
T	F	T		T	T	**T**		F	
T	F	F		F	F	**T**		T	
F	T	T		T	T	**T**		F	
F	T	F		T	T	**T**		F	
F	F	T		T	T	**T**		F	
F	F	F		T	F	**T**		T	

An example of a contradiction is $(P \Rightarrow Q) \wedge (P \wedge \neg Q)$:

P	Q	$(P \Rightarrow Q)$	$\wedge$	$(P \wedge \neg Q)$
T	T	T	**F**	F
T	F	F	**F**	T
F	T	T	**F**	F
F	F	T	**F**	F

3 Set Theory

Next, we wish to talk about **sets** of objects in the world. The way we group object into sets is by using logic. Sets are defined in two ways, **intensional** and **extensional**. For extensional, we explicitly list all the elements of a set

$$S = \{yellow, green, blue\}.$$

For intensional, we say that each element must make a statement about it true.

$$S = \{x : x \text{ is one of the 50 states that starts with the letter N}\}.$$

To denote **membership**, we use $x \in S$ to say x is an element of set S. We can describe new sets using five main symbols: $\cup$ for **union**, $\cap$ for **intersection**, $'$ for **complement**, $\subset$ for **subset** and $=$ for **equality**.

$$S \cup T = \{x : x \in S \vee x \in T\}.$$

$$S \cap T = \{x : x \in S \wedge x \in T\}.$$

$$S' = \{x : \neg(x \in S)\}.$$

$$S \subset T \text{ if for all } x \in S, x \in T.$$

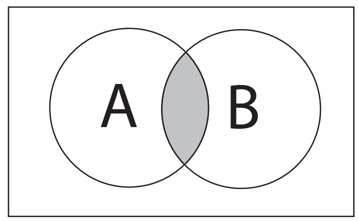
$$S = T \text{ if } S \subset T \wedge T \subset S.$$

One last set is $\emptyset$, the **empty set**, which has no elements. $S \cap S' = \emptyset$.

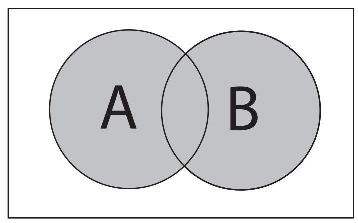
4 Venn Diagrams

Often, it is easier to visualize the membership of a set using a **Venn diagram**, as shown below.

A intersects B



A unioned with B



5 Counting Sets

As we begin probability, it is important to know the **size** of a set, and to be able to calculate the size of other sets. We denote the size of set S as $|S|$. Set size obeys the following properties:

$|S \cup T| = |S| + |T| - |S \cap T|$.

$|S \cap T| = |S| + |T| - |S \cup T|$.

If $S \subset U$, then $|S'| = |U| - |S|$.

$|S \times T| = |S| \times |T|$.

6 Basic Probability

One way to calculate probability is through **relative frequency**, which involves performing many experiments and counting the results.

$$P(event) = \frac{number\ of\ desired\ outcomes}{total\ possible\ outcomes} \tag{1}$$

If we make some general assumptions about how the world works, such that we make choices uniformly at random, we can use set theory to find probabilities. If $S \subset U$, then

$$P(A) = \frac{|A|}{|U|} \tag{2}$$

The **multiplication rule** is helpful if we know two events are **independent** (do not affect each other).

$$P(A \cap B) = P(A) \times P(B) \tag{3}$$

The **addition rule** is helpful if we know two events are **mutually exclusive** (the sets have no overlap).

$$P(A \cup B) = P(A) + P(B) \tag{4}$$

A few more probability rules, where U is the set of all possibilities, and $A \subset U$.

$$P(U) = 1$$

$$P(\emptyset) = 0$$

$$P(A) = 1 - P(A')$$

To calculate if an event with probability p will occur *at least once* in n trials, we use

$$1 - (1 - p)^n \tag{5}$$

which is another way of saying "1 - the chance of the even *never* occuring."

7 Combinatorics

To answer the question of how many ways are there to put n items in order, we need the **factorial**.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \tag{6}$$

For our first item, we have n choices, but once we remove that item, we have only $n-1$ choices for the second item. This continues decreasing by one until we have only 1 item left at the end.

With the factorial, we can define **permutations**, which answer questions such as "How many orderings can we create if we select k items from a set of size n?"

$$\frac{n!}{(n-k)!} \tag{7}$$

If we further do not care about the ordering of the subsets, only the elements, we can define **combinations** as $\binom{n}{k}$, which is read "n choose k".

$$\binom{n}{k} = \frac{n!}{(n-k)!n!} \tag{8}$$

These combinations have some interesting properties:

$$\binom{n}{k} = \binom{n}{n-k} \tag{9}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{10}$$

This is the idea behind **Pascal's triangle**, where each number is the sum of the two numbers above it.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
. & . & . & . & . & . & . & . & .
\end{array}$$

Using combinations, we now have a tool to calculate the probability that an event with probability p will occur exactly k times in an experiment repeated n times.

$$\binom{n}{k} p^k (1-p)^{n-k} \tag{11}$$

8 Calibration

When presented with a weighted die or coin, such that the probabilities will differ from what we expect from a fair coin, we can quickly create a way to **recalibrate** the coin flips or die rolls to be fair. With a coin, we flip it multiple times, then ask if the total number of Heads was odd or even. If d is the difference between our weighted coin and the expected outcome, then if we flip the coin n times, our new probability will be

$$\frac{1+d^k}{2} \tag{12}$$

In general, we use **modular division** if we have more than two options. With n options, total up the results to be t, then our fair result will be $(t\%n) + 1$, where % calculates the remainder of division. Our recalibrated probabilities will be

$$\frac{1 + d^k}{n} \tag{13}$$

9 Conditional Probability

If we already know the outcome of some experiments, we say our knowledge is **conditioned** on this information. Given we know B, what is the probability of A?

$$P(A|B) = \frac{P(A \cap B)}{P(A)} \tag{14}$$

In essence, we are changing our world to only B, and calculating the relative size of $A \cap B$.

10 Bayes' Rule

Often times, we know $P(B|A)$, but really wish to know $P(A|B)$. If we know that certain diseases cause symptoms, but a patient only displays symptoms, and we wish to know if they indicate a certain disease. As long as we know both $P(A)$ and $P(B)$, we can calculate with **Bayes' rule**.

$$P(A|B) = \frac{P(B|A) \times P(B)}{P(A)} \tag{15}$$