Math 105: Finite Mathematics

Study Guide 2

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1 Odds

Probabilities measure the number of desired outcomes relative to the number of possible outcomes. **Odds** measure the number of desired outcomes in a ratio to the number of undesired outcomes.

They say even a stopped clock is correct twice a day. There are 60 minutes each hour and 24 hours in a day, for 1440 possible values of time. So the stopped clock has a $\frac{2}{1440}$ probability of being correct, reduced to $\frac{1}{770}$.

However, the odds of the stopped clock being correct are expressed as 1:769, since there are 2 desired outcomes and 1438 undesired outcomes.

A casino may tell you the **Monetary** odds, which is how much they will pay if you win, compared to how much you owe if you lose. These are usually a little worse than the true odds of winning, such that they will not pay as much as you should be earning.

2 Expected Value

The expected value is an extension of the average value, where each value is not equiprobable.

To average a set of numbers, you add them up and divide by the total. So the average of $\{3, 4, 8, 7\}$ would be $\frac{3+4+8+7}{4} = \frac{22}{4} = 5.5$. This is the same as $3 * \frac{1}{4} + 4 * \frac{1}{4} + 8 * \frac{1}{4} + 7 * \frac{1}{4} = \frac{3}{4} + \frac{4}{4} + \frac{8}{4} + \frac{7}{4} = 5.5$, where $\frac{1}{4}$ is the probability of a number from this set being selected randomly.

Expected value for a random value X follows this same pattern, but the probability of selecting each value is not always uniform. In mathematical terms

$$E(X) = \sum_{i=0}^{n} P(X_i) * v_i$$

where there are n possible worlds, $P(X_i)$ is the probability of being in world i and v_i is the value of being in world i. This is the sum of each value time the probability of seeing that value.

To find the expected value:

- Find the possible worlds (X_i)
- Find the value of ending up in each world (v_i)
- Find the probability of ending up in each world ($P(X_i)$)
- Multiply each $P(X_i)$ times v_i, and sum them up.

For example, a "friend" offers you the following bet, if you flip a coin three times, and exactly two of those flips are heads, then he will pay you $1, otherwise you will owe him $1.

There are two possible worlds, either you have exactly two heads, or you do not. There are 8 possible ways the coins could be flipped, and three of those (HHT, HTH, and THH) will earn you $1, the other five will make you lose $1. So the expected value of this bet is

$\frac{3}{8} * 1 + \frac{5}{8} * -1 = -0.25$

You should expect to lose 25 cents each game if you play this game repeatedly.

3 Standard Deviation

The standard deviation is used to describe the spread of data, or the risk involved in the expected value from a bet. If the expected values are equal and you are less inclined to be risky, you should take a bet with a lower standard deviation.

You calculate the standard deviation by finding the square root of the Expected Value of the divergence from the mean squared.

$$st.\ dev. = \sqrt{E((X - E(X))^2)}$$

To find the standard deviation:

- Find the Expected Value ($E(X)$, see above section)
- Subtract the value of ending up in each world (v_i) from the Expected Value ($E(X)$) We will call this $delta_i$.
- Square these new $delta_i$ values

- Multiply the probability of ending up in each world ($P(X_i)$) by the $delta_i$ for that world, and sum them up

- Take the square root of the final Expected Value

For example, we will look at the standard deviation of our game above. The expected value is -0.25. So our individual $delta_i$ values will be 1 - -0.25 = 1.25 and -1 - -.25 = -0.75. These values squared are 1.5625 and -0.5625. We calculate the expected value for these numbers,

$\frac{3}{8} * 1.5625 + \frac{5}{8} * -0.5625$ to get 0.234375. Now the square root of this is 0.48412, our standard deviation.

4 Binary Numbers

We are most familiar with decimal numbers written in base 10 notation, such that there is a ones place, tens place, hundreds place, etc. The language of computers, and some other branches of mathematics, is in binary numbers. Instead of each digit being one of the numbers {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, our choices are now {0, 1}. This means our digit places are 1, 2, 4, 8, 16, etc.

To convert a decimal number n to a binary number

- Find the largest power of two (p) which is less than or equal to the number

- Now repeat the following steps while the number $p >= 1$:

A: If $p <= n$, write a 1, otherwise a 0, then subtract p from n

B: Divide p by 2

For example, we will convert the number 47. 32 is the largest power of 2 smaller than 47, so $p = 32$.

32 < 47, so write a 1

n now equals 47 - 32 = 15

p now equals 32 / 2 = 16

16 is not < 15, so write a 0

p now equals 16 / 2 = 8

$8 < 15$, so write a 1

n now equals 15 - 8 = 7

p now equals 8 / 2 = 4

$4 < 7$, so write a 1

n now equals 7 - 4 = 3

p now equals 4 / 2 = 2

$2 < 3$, so write a 1

n now equals 3 - 2 = 1

p now equals 2 / 2 = 1

$1 < 2$, so write a 1

n now equals 1 - 1 = 0

p now equals 1 / 2 = 0.5, so we stop.

Our resulting binary number for 47 is 101111

5 Nim (Ch. 21)

The game of Nim has two players competing to be the last player to make a valid move. There are n piles of stones, and a move is to take as many stones as you want from one pile.

You can either win or lose the game of Nim, there is no draw. We can determine if the first player or the second player will win, given that both players make the optimal move, by examining the sum of the piles using binary numbers.

First, convert the size of each pile into a binary number. We then add each of these number, but instead of carrying the digits, we only worry if there are an even or odd number of 1s. If the number of 1s is odd, the sum has a 1 in this place. If the number of 1s is even, the sum is a 0 in this place.

If the nim-sum of the binary numbers is zero, then the second player can win, otherwise it is possible for the first player to win.

When the num-sum is not zero, we wish to know what move we should make to win the game. This will be the move that make the nim-sum zero. To find this number, we add the nim-sum to the binary number for each pile using nim addition. The answer which is less than the current pile size will be the correct move to make.

For example, we have a game with two piles, size 12 and 5. In binary, 12 is 1100, and 5 is 0101. Their nim-sum is 1001, so it is possible for the first player to win the game. For 12, 1100 + 1001 is 0101, or 5, and for 5, 0101 + 1001 is 1100, or 12. So our optimal move is to take 7 stones from the 12 pile, leaving the other player with two piles of 5 stones each.

6 Combinatorial Games (Ch. 18, 19, 27)

With abstract, two-player games, where there is no randomness involved, and each move makes progress toward the end of the game, there are only three possible outcomes. Either the first player wins, the second player wins, or there is a draw.

If we say that the first player prefers winning states (relative to the first player) over draws, and draws over losing states (relative to the first player), and the second player prefers losing states (relative to the first player) over draws, and draws over winning states (relative to the first player), and assume that each player will play optimally, we can determine from the initial board position if the game should be won by the first or second player or if it will end in a draw.

First, we draw the complete game tree. From each board state, we draw children for each possible move. This is done until we reach the leaves, which are states that are either wins, losses or draws.

Now, we can propagate this information upwards in the tree from the leaves to the root. For each non-leaf node, we examine the children. If it is the second player's turn to move, label this internal node with the lowest child's value, choosing Loss over Draw over Win. If it is the first player's turn to move, label this internal node with the highest child's value, choosing Win over Draw over Loss. If we start at the lowest level and move up, we will eventually reach the initial state, and be able to determine what will happen with optimal play from both players.